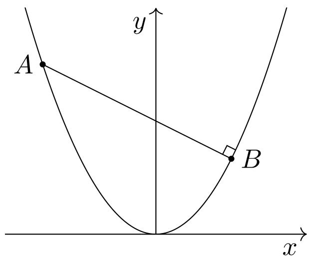
4201. Show that, when three dice are rolled together, the probability that two of the scores will add up to the other is $5/24$.

4202. A chord AB is drawn to the parabola $y = x^2$. At B, coordinates $\left(p, p^2\right)$ in the positive quadrant, the chord is normal to the curve.



Prove that $|AB| > 2p$.

4203. Functions f and g are defined over the reals, and have ranges $[a, c]$ and $[b, d]$ respectively, where

$$a < b < c < d.$$

For each of the following functions defined over $\mathbb{R}$, give the smallest set which can be guaranteed to contain the range:

(a) $x \mapsto \text{f}(x) + \text{g}(x)$,

(b) $x \mapsto \text{f}(x) - \text{g}(x)$.

4204. Consider the graph $y = \log_a x$, for $a > 0$.

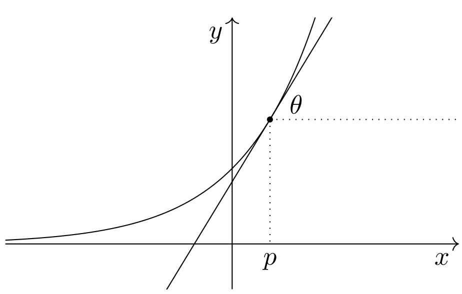
(a) Sketch the cases $a = 2, 3$ on the same axes.

(b) Prove that $y = \log_a x$ is a scaled version of $y = \log_b x$, and determine the scale factor and direction of the relevant stretch.

4205. Show, by integration, that

$$\int \frac{3x - 6}{\sqrt{2x + 3}}\, dx = (x - 9)\sqrt{2x + 3} + c.$$

4206. The exponential graph is $y = e^x$. At $x = p$, the graph is at inclination θ above the x axis.



Show that $\sin^2\theta = \dfrac{e^{2p}}{e^{2p} + 1}$.

4207. Prove, by contradiction, that $\sqrt[3]{2}$ is irrational.

4208. A particle is modelled as having an acceleration proportional to the square of its velocity.

(a) Show that this information can be written, for some constant $k \in \mathbb{R}$, as

$$\int \frac{1}{v^2} \frac{dv}{dt}\, dt = \int k\, dt.$$

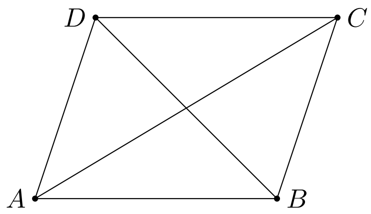
(b) Hence, show that $v = \dfrac{1}{c - kt}$.

4209. Find the set of values $x \in [0, 2\pi)$ which satisfy

$$\sec x < \operatorname{cosec} x.$$

4210. The graphs $y = 4x^2 - 2$ and $y = x^4 + k$ are tangent. Determine all possible values of the constant k.

4211. Prove that, if the diagonals of quadrilateral $ABCD$ bisect each other, then $ABCD$ is a parallelogram.



4212. From a computer simulation of a random process, the following probabilities are known:

$$\mathbb{P}(A) = 0.2,$$
$$\mathbb{P}(A \cap B) = 0.1,$$
$$\mathbb{P}(A \cap B \cap C) = 0.05.$$

For each of the following, find the probability or else state that it cannot be ascertained from the information given.

(a) $\mathbb{P}(A \mid B)$,

(b) $\mathbb{P}(B \mid A)$,

(c) $\mathbb{P}(A \cap C)$,

(d) $\mathbb{P}(B \cap C \mid A)$.

4213. Determine the range of $\text{f}(x) = \left(1 + x + x^2\right)^4$, when f is defined over the real numbers.

4214. With θ in radians, a function has instruction

$$\text{f}(\theta) = \frac{\cos\theta}{1 + \theta^2}.$$

This function is not integrable. By writing $\text{f}(\theta)$ as a quadratic, determine the approximate value of

$$\int_0^{\frac{\pi}{36}} \text{f}(\theta)\, dx.$$

4215. A curve is defined implicitly by

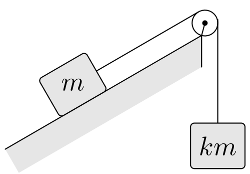
$$(x-k)(y-k)+x^2y^2=0.$$

Show that, when k is large and positive, the curve has stationary points at $y \approx \pm\frac{1}{2}$.

4216. State, with a reason, a general formula for

$$\int \frac{\mathrm{f}'(x)}{(a\,\mathrm{f}(x)+b)^2}\,dx.$$

4217. A pulley system is set up as below on a rough slope. The slope is inclined at $\theta = \arcsin\frac{3}{5}$ above the horizontal. The coefficient of friction on the slope is $\frac{1}{4}$. The system remains in equilibrium.



Making any necessary assumptions, determine the set of values of the ratio k for which the system will remain in equilibrium.

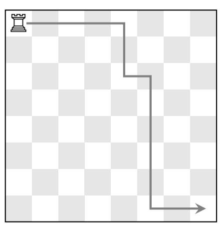
4218. Prove that $\displaystyle\sum_{i=1}^{n}\left(\sum_{j=1}^{n} ij\right) \equiv \frac{1}{4}n^2(n+1)^2$.

4219. Solve the following simultaneous equations:

$$\begin{aligned}&\log_2 x + 2\log_4 y = 1\\ &x+y=3.\end{aligned}$$

4220. Show that the curves $y=e^x$ and $y=e^x\cos x$ are tangent to each other at infinitely many points.

4221. On an 8×8 chessboard, a rook may move any number of squares along a single row or column. A rook starts at the top-left corner, and moves to the bottom-right corner.



Show that, if every move is either downwards or rightwards, then there are 3432 distinct paths it can take.

4222. Find, to three significant figures, the area enclosed by the x axis and the octic curve

$$y = 1.5236x^8 - 2.0767x^3 - 6.8814.$$

4223. Find k such that the following equation has exactly two real roots:

$$x^2 - \frac{1}{2x+k} = 0.$$

4224. You are given that, for a function f and starting value x_0, the following iteration converges to α:

$$x_{n+1} = x_n - \frac{\mathrm{f}'(x_n)}{\mathrm{f}''(x_n)}.$$

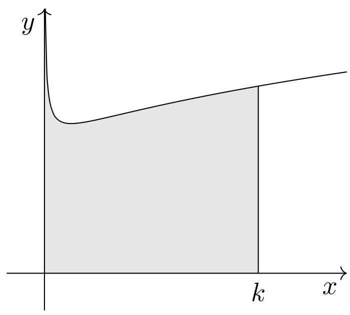
State, with a reason, the value of $\mathrm{f}'(\alpha)$.

4225. Provide counterexamples to the following claims:

(a) "Two cubic graphs of the form $y=\mathrm{f}(x)$ and $y=\mathrm{g}(x)$ must intersect."

(b) "Every graph of the form $y=x^{2k+1}$, for $k\in\mathbb{Z}$, intersects the x axis at least once."

4226. The diagram shows a region of area 18 square units, whose boundaries are the axes, the line $x=k$ and the curve $y=4x^{\frac{1}{3}}+2x^{-\frac{1}{3}}$.



Determine the exact value of k.

4227. A cube-root surd expression is given by

$$\frac{1}{\sqrt[3]{2}+1}.$$

A student tries to rationalise the denominator, by using the standard square-root technique.

(a) Show that multiplying the denominator by $\sqrt[3]{2}-1$ will not rationalise it.

(b) The correct factor is $2^{\frac{2}{3}}-2^{\frac{1}{3}}+1$. Rationalise the denominator of the surd.

4228. (a) Prove that, for a pair of parametric equations, the second derivative is given by

$$\frac{d^2y}{dx^2} \equiv \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

(b) Determine the coordinates of the two points of inflection of the curve $x=t^2$, $y=t+t^3$.

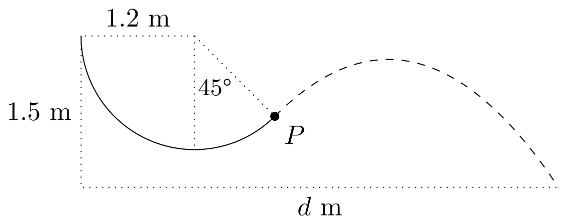
4229. Points A, B, for some constant k, have coordinates

$$A : (e^k, e^k)$$
$$B : (e^{-k}, -e^k).$$

You are given that $\overrightarrow{AB}$ is parallel to the vector $\mathbf{i} + 3\mathbf{j}$. Show that $k = \ln\sqrt{3}$.

4230. Three numbers are chosen from the set $[0, 1]$. Find the probability that these sum to less than 1.

4231. A small metal ball is released on a track in the shape of a circular arc. The ball leaves the track at point P, travelling at 2 ms^{-1}, and becomes a projectile.



Determine d.

4232. A family of functions is defined, over the largest possible real domain, as

$$\mathrm{A}_n(x) = \frac{nx+1}{nx-1}.$$

(a) Solve $\mathrm{A}_1(x) + \mathrm{A}_2(x) = 0$.

(b) Assuming x is small, give an approximation for $\mathrm{A}_1(x)\,\mathrm{A}_2(x)$ in the form $b_0 + b_1x + b_2x^2$, where the constants b_0, b_1, b_2 are to be determined.

4233. Consider the graph of $y = \dfrac{6x+1}{2x-9}$.

(a) The lines $x = a$ and $y = b$ are asymptotes of the graph. Find the value of a and of b.

(b) Show that the graph has no stationary points.

(c) Hence, sketch the graph.

4234. Solve the equation $2^{\sin x} = 4^{\cos x}$, for $x \in [0, \pi)$.

4235. A quartic and a line have equations

$$y = x^4 - 2x^3,$$
$$y = -2x + 1.$$

(a) Show that the line is a tangent to the quartic.

(b) Show that the line crosses the quartic at its point of tangency.

4236. Show that $\tan\frac{5\pi}{12} = 2 + \sqrt{3}$.

4237. A door of weight 200 N, measuring 110 cm by 230 cm, is modelled as a uniform rectangular lamina in a vertical (x, z) plane. It is freely hinged along the z axis. Two people exert forces on opposite sides of the door. One pushes with 100 N at $(80, 115)$ in the positive y direction, the other pushes with 125 N at $(64, 115)$ in the negative y direction.

(a) Draw a force diagram in the (x, y) plane.

(b) Show that the door remains in equilibrium.

(c) The hinges exert a total force $\mathbf{R}$ on the door.

i. Find the magnitude of the component of $\mathbf{R}$ in the (x, y) plane,

ii. Find the total contact force $\mathbf{R}$, giving your answer as a column vector.

4238. Prove that, if $y = \mathrm{h}(x)$ has rotational symmetry, then so does $y = \mathrm{h}''(x)$.

4239. To find, from first principles, the gradient of the curve $y = x^{\frac{1}{3}}$, the following limit is set up

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h}.$$

(a) Find and simplify the first three terms of the binomial expansion of

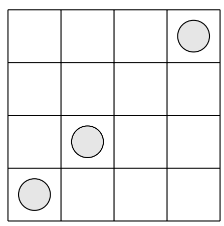
$$\left(1 + \frac{h}{x}\right)^{\frac{1}{3}}$$

(b) Hence, or otherwise, prove that

$$\frac{dy}{dx} = \tfrac{1}{3}x^{-\frac{2}{3}}.$$

4240. Prove that $a = b^4 \implies a > b^2 - 4$.

4241. Three counters are to be placed on a four-by-four grid, with a maximum of one per grid square, such that they are collinear.



Show that there are 44 distinct ways in which this may be done.

4242. You are given that $\arctan x - kx = 0$, where k is a real constant, has precisely one root. Find all possible values of k, answering in set notation.

4243. Independent random variables X_1 and X_2 each have the distribution $X_i \sim N(0, 1)$. Find

$$\mathbb{P}(X_1 < 1 - X_2).$$

4244. You are given that, for some constants $a, b, c \in \mathbb{R}$,

$$ax + by = c.$$

Prove that the least possible value of the quantity $x^2 + y^2$ is

$$\frac{c^2}{a^2 + b^2}.$$

4245. Show that the shape enclosed by the two graphs $x = |y|$ and $x = 4 - \frac{1}{2}|y|$ is a kite.

4246. In a model, a particle moves in a hypothetical four-dimensional space, with position vector given by

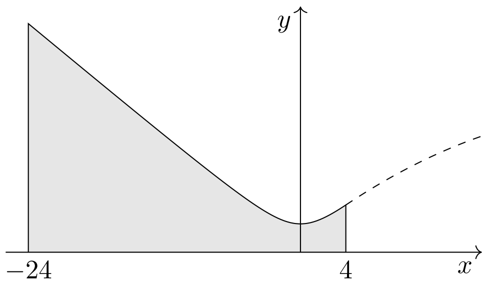
$$\mathbf{r} = \begin{pmatrix} 1 + 2t \\ 2 - 4t^2 \\ t \\ 4 + 3t^2 \end{pmatrix}$$

Show that the particle's path is parabolic.

4247. Sketch $y^2 = \sin x \cos x$ for $x \in [0, 2\pi]$.

4248. Take g to be 10 in this question.

A particular ski-jump is designed to follow the curve $4y^2 - x^2 = 9$, for $x \in [-24, 4]$. The jumper is modelled as a particle, and lengths are measured in metres.



A ski-jumper leaves the slope at $x = 4$, with speed $5\sqrt{29}$ ms^{-1}. Show that, ignoring air resistance, the equation of the subsequent trajectory is

$$250y = 193 + 116x - 2x^2.$$

4249. In the nth row of Pascal's triangle, the value in position r is notated ${}^n\mathrm{C}_r$, where both n and r are defined from the zeroth position. By simplifying ${}^n\mathrm{C}_r + {}^n\mathrm{C}_{r+1}$, show that the formula

$${}^n\mathrm{C}_r = \frac{n!}{r!(n-r)!}$$

satisfies the addition property of Pascal's triangle.

4250. You are given that

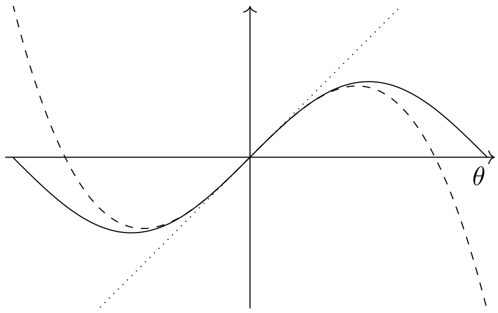
$$\tan 10° \times \tan 50° = \tan 20° \times \tan 30°.$$

Show that

$$\tan 10° = \tan 20° \times \tan 30° \times \tan 40°.$$

4251. This question concerns the cubic approximation to the sine function, defined in radians, for small θ:

$$\sin\theta \approx a\theta + b\theta^2 + c\theta^3.$$



(a) Write down the value of a.

(b) Using the second derivative, find b.

(c) Determine the value of c.

4252. Solve $\dfrac{1}{1+2x} + \dfrac{1}{1-2x} + \dfrac{1}{2+x} + \dfrac{1}{2-x} = 0.$

4253. Function A has instruction

$$\mathrm{A} : k \longmapsto \int_0^1 x^{-k}\,dx.$$

The domain is a subset of $\mathbb{R}$; the codomain is $\mathbb{R}$.

(a) Show that $\mathrm{A}(k)$ is well defined for $k \in (0, 1)$.

(b) Show that $\mathrm{A}(k)$ is not well defined for $k = 1$.

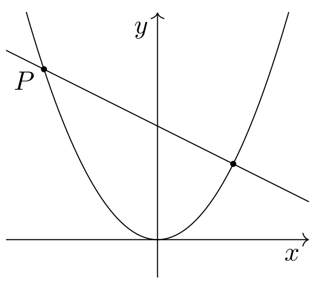
4254. A polynomial function g has

$$\mathrm{g}(\alpha) = \mathrm{g}'(\alpha) = \mathrm{g}''(\alpha) = 0.$$

Prove that $\mathrm{g}(x)$ has a factor of $(x - \alpha)^3$.

4255. Write down the value of $\left(\operatorname{cosec}\frac{\pi}{6}\right)^{-2} + \left(\sec\frac{\pi}{6}\right)^{-2}$.

4256. A normal is drawn to the curve $y = x^2$. It crosses the curve again at point P.



Show that the y coordinate of P is at least 2.

4257. Starting from the formula for $\tan(\theta + \phi)$, prove that, if $x = \tan\theta_1$, $y = \tan\theta_2$ and $z = \tan\theta_3$, then

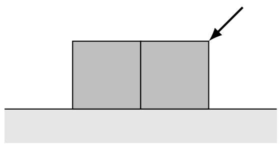
$$\tan(\theta_1 + \theta_2 + \theta_3) \equiv \frac{x + y + z - xyz}{1 - (xy + yz + xz)}.$$

4258. Two blocks of wood, each of weight 5 N, sit on a rough tabletop. The coefficient of friction between each block and the tabletop is μ, and the contact between the blocks is smooth.

A child pushes one block diagonally downwards at $45°$ to the tabletop, with a force of $2\sqrt{2}$ N. The system of the two blocks is on the point of sliding.

(a) Determine the smallest possible value of μ, the coefficient of friction.

(b) Show, by giving two explicit examples, that μ cannot be determined from the information in the question.



4259. A curve has equation $y = \tan x - 2\sec x$, defined for $x \in [0, 2\pi]$.

(a) Show that this curve does not cross the x axis.

(b) Find and classify all stationary points.

(c) Find any asymptotes.

(d) Hence, sketch the curve.

4260. The parabola $y = ax^2 + bx + c$ has x intercepts at $x = p, q$. Find, with coefficients in terms of a, b, c, the equation of the monic cubic that passes through the origin and has SPs at $x = p, q$.

4261. A family of circles is defined, for $k = 0, 1, ..., 5$, by

$$\left(x - 2\cos\frac{k\pi}{3}\right)^2 + \left(y - 2\sin\frac{k\pi}{3}\right)^2 = 1.$$

Sketch the family of circles, making any points of tangency clear.

4262. In this question, do not use a polynomial solver.

The quartic equation $8x^4 - 4x^3 - 6x^2 + 5x - 1 = 0$ has a triple root.

(a) Explain how you know that the equation must also have a single root.

(b) Expand and simplify $a(x + b)^3(x + c)$.

(c) Hence, determine the roots of the quartic.

4263. You are given that, for a particular polynomial function f, and for **all** real values of k,

$$\int_0^k \mathrm{f}(x)\,dx + \int_{-k}^0 \mathrm{f}(x)\,dx = 0.$$

Show that $y = \mathrm{f}(x)$ has rotational symmetry.

4264. A function is defined over $\mathbb{R}$ as

$$\mathrm{g}(x) = e^{ax} - e^{bx}.$$

Determine the conditions on the constants a, b for there to be a stationary value of the function g.

4265. A particle has position vector, at time t, given by

$$\mathbf{r} = \cos 2t\mathbf{i} + \sin 2t\mathbf{j} + 5t\mathbf{k}.$$

(a) Show that the particle stays on the surface $x^2 + y^2 = 1$, and describe this surface.

(b) Show that speed is constant.

(c) Describe the motion.

4266. New coordinate variables X and Y are defined, in terms of the usual variables x and y and constants a and b, by

$$X = \frac{ax + by}{\sqrt{a^2 + b^2}}, \quad Y = \frac{bx - ay}{\sqrt{a^2 + b^2}}.$$

(a) The (x, y) point $(1, 0)$ is converted to the new coordinates. Show that the resulting (X, Y) point lies at unit distance from the origin.

(b) By solving as a pair of simultaneous equations, find simplified expressions for x and y in terms of a, b, X, Y.

4267. A die is rolled six times. Find the probability that three different scores each appear twice.

4268. In the game of tic-tac-toe, two players alternately place ○ and × in a three-by-three grid. A player who places three in a row wins. So, in the example game below, ○ has won:

○	×	
○	○	×
○		×

Show that, if the first player plays in the centre, then the second player must play in the corner.

4269. A function E, in the form of an infinite series, is defined over the domain $\mathbb{R}$ by

$$\mathrm{E}(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

You are given that $\mathrm{E}(1) = e$.

(a) Assuming that the series converges, show that

$$\mathrm{E}'(x) = \mathrm{E}(x).$$

(b) By separation of variables, solve to find the particular solution of this differential equation.

(c) Hence, show that $\dfrac{1}{5!} + \dfrac{1}{6!} + \dfrac{1}{7!} + ... < \dfrac{1}{100}$.

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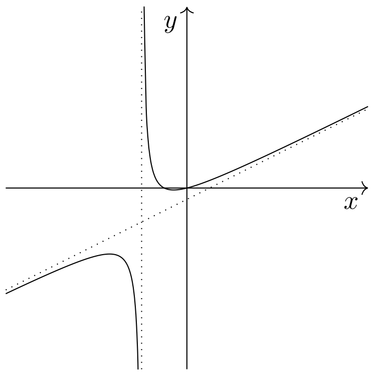
4270. Sketch $y = |x-1| \times |x-2| \times |x-3|$.

4271. The curve shown below has a vertical asymptote at $x = -1$ and an oblique asymptote at $4y = 2x - 1$.



The equation of the curve is

$$y = \frac{ax^2 + bx}{cx + 4}.$$

Find the values of the constants a, b, c.

4272. Solve, for $x, y \in [0, 2\pi)$,

$$\sqrt{2}\sin x - 2\cos y = 0,$$
$$\sqrt{2}\cos x + 2\sqrt{3}\sin y = 4.$$

4273. Prove that, given the first and any other term of a geometric sequence, any term of the form u_{2k+1}, for $k \in \mathbb{N}$, can be calculated with certainty.

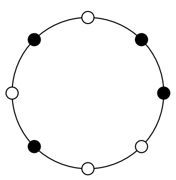
Note: in this question, "given" means given both the location and value of the term, e.g. $u_7 = 31$.

4274. A vector is given, for $t \in \mathbb{R}$, by

$$\mathbf{b} = (\tan^2 t)\mathbf{i} + (\sqrt{3}\sec t)\mathbf{j}.$$

Show that $|\mathbf{b}|$ is never less than $\sqrt{3}$.

4275. A set of eight beads, four black and four white, is threaded onto a circular bracelet.



Determine the number of possible arrangements, if rotations and reflections are not counted as distinct.

4276. The curve $y = x^2 e^x$ has two stationary points.

(a) Find and classify the stationary points.
(b) Hence, sketch the curve.

4277. A ladder of mass m is standing on rough ground, with coefficient of friction μ, leaning up against a smooth vertical wall.

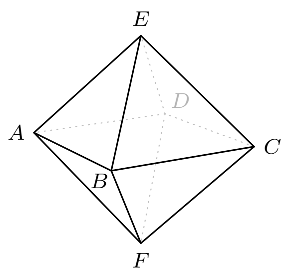
(a) Find, in terms of μ, the greatest angle to the vertical at which the ladder can be placed without it slipping.
(b) At this angle, determine the mass which would have to be fixed to the bottom of the ladder to allow a person of mass M to climb all the way to the top.

4278. Sketch the following curve:

$$y = x^2 + \frac{x-1}{x+1}.$$

You don't need to find SPs or intercepts.

4279. A regular octahedron is shown below. An ant is walking the surface of the octahedron, starting at A. It chooses an edge at random and walks its length. Upon reaching another vertex, it chooses a new edge at random, never leaving the edges and never travelling the same edge twice.



The ant has walked exactly four edges. Find the probability that it is back at point A.

4280. A differential equation is given as

$$\frac{dx}{dt} - \frac{2x}{t} + t^2x^2 = 0.$$

Verify that the following is a solution curve:

$$x = \frac{5t^2}{t^5 + c}.$$

4281. A curve is defined implicitly as $x^2 + xy + y^2 = 1$.

(a) Express $x^2 + xy + y^2$ in terms of the variables

$$p = x + y,$$
$$q = x - y.$$

(b) Hence, show that the curve is an ellipse.

4282. In this question, neglect air resistance.

Prove that, if you hold on to a rising balloon, thus accelerating upwards at a constant rate from rest at ground level, then landing speed is proportional to the length of time before you let go.

4283. The discriminants of positive quadratic functions f and g are zero and positive respectively. For each of the following graphs, write down the number of x intercepts and vertical asymptotes:

(a) $y = \dfrac{f(x)}{f(x) - 1}$,

(b) $y = \dfrac{g(x)}{g(x) - 1}$,

(c) $y = \dfrac{f(x)}{g(x)^2 + 1}$.

4284. An equilateral triangle is centred on the origin of an (x, y) plane, with one of its vertices at $(1, 0)$. A second triangle is then drawn, rotated by 30° around the origin.

(a) Show that, at three points of intersection, the triangles intersect at right angles.

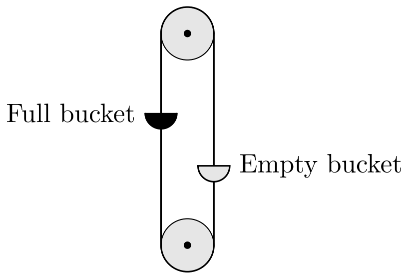
(b) Show that the other points of intersection lie at a distance $\frac{1}{2}\operatorname{cosec} 75°$ from the origin.

4285. Prove that, for $p, q \in \mathbb{R}$,

$$p > q \implies p^3 - p^2 + p > q^3 - q^2 + q.$$

4286. Sketch $y = \sin^4 x$.

4287. In a chemical factory, a fan-belt moves vertically over two fixed pulley wheels, filling, raising and emptying a pair of buckets. At any time, one bucket is full and the other is empty.



The fan-belt is modelled as light and inextensible, and each bucket has mass 10 kg. A driving force of $8g$ N is exerted on the belt by the pulley wheels, which keeps the belt moving at constant speed. Air resistance is negligible.

For each of the following quantities, determine the value or explain why it cannot be determined given the information in the question:

(a) the tension below the full bucket,

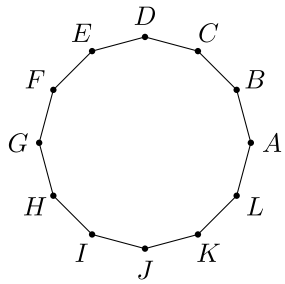
(b) the tension above the full bucket,

(c) the mass of liquid carried by a full bucket.

4288. Solve the equation $t^{\frac{1}{3}} = 2 + 15t^{-\frac{1}{3}}$.

4289. Show that the graphs $x^4 + y^2 = 1$ and $x^2 + y^2 = 1$ have exactly four points of tangency.

4290. A regular dodecagon has twelve vertices, which are labelled alphabetically.



Find $\angle BFK$.

4291. The function $h(x) = x^2 + |x| - 6$ is defined over $\mathbb{R}$. Solve the equation $h(x) = 0$.

4292. Sketch $xy + \dfrac{1}{xy} = 4$.

4293. A two-tailed binomial hypothesis test is set up, with null hypothesis $H_0 : p = 0.34$.

(a) Write down the alternative hypothesis.

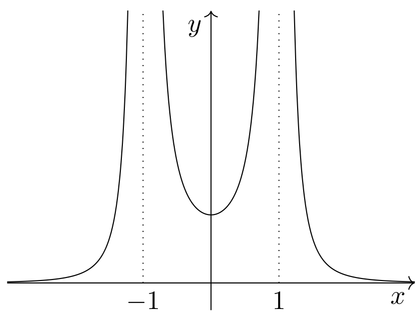
(b) Find the critical region, at the 5% significance level, for a sample of 50. Give your answer in set notation.

(c) Write down the conclusion of the test for a sample of 50 with $x = 11$.

(d) A second sample of 50 is taken, which also yields $x = 11$. Explain why this sample might warrant a different conclusion.

4294. Show that $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x \, dx = 1 - \tfrac{\pi}{4}$.

4295. A student is asked to sketch $(x^2 - 1)\sqrt{y} = 1$. He produces the following graph:



Explain and correct the error in his answer.

4296. A sequence is given, for constants a, b, by

$$x_{n+1} = \frac{a}{b - x_n}.$$

Prove that, if the sequence is to exhibit period 2 behaviour, then either $b = 0$, or $b^2 \geq 4a$.

4297. True or false?

(a) The graph $x^4 + y^4 = 1$ is bounded,
(b) The graph $x^5 + y^5 = 1$ is bounded,
(c) The graph $x^6 + y^6 = 1$ is bounded.

4298. A graph is given, for constants $A, B, k_1, k_2 \in \mathbb{N}$, by

$$y = A \cos k_1 x + B \sin k_2 x.$$

Prove that the graph has infinitely many SPs.

4299. Show that the normal to $y = x^2$ at $x = p$ crosses the curve again at

$$y = \frac{(2p^2 + 1)^2}{4p^2}.$$

4300. Relation R is given implicitly as

$$|y + x - 1| + |y - x| = 1.$$

Prove that the locus of R is a unit square.

——— End of 43rd Hundred ———